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On the improvement of analytic properties under the limit q-Bernstein operator

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Dedicated to the memory of my colleague and friend Yurij Kirichenko

Abstract

Let $B_n(f, q; x)$, $n = 1, 2, \ldots$ be the q-Bernstein polynomials of a function $f \in C[0, 1]$. In the case $0 < q < 1$, a sequence $\{B_n(f, q; x)\}$ generates a positive linear operator $B_\infty = B_{\infty,q}$ on $C[0, 1]$, which is called the *limit q-Bernstein operator.* In this paper, a connection between the smoothness of a function f and the analytic properties of its image under B_∞ is studied.

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1. Introduction

Since Bernstein polynomials play an important role in Approximation Theory and its applications, their various generalizations have been studied (see, e.g. [5,10,17,4,13]).

Due to the intensive development of q-Calculus, the generalizations of Bernstein polynomials connected with q-Calculus have emerged.

In 1997, Phillips [14] introduced the q-Bernstein polynomials. While for $q = 1$ these polynomials coincide with the classical ones, for $q \neq 1$ we obtain new polynomials with interesting properties. These polynomials have been studied lately by a number of authors, see [3,7,9,11,15]

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and references therein, [18–21]. The convergence of q-Bernstein polynomials was considered in [9,11,18–21]. In [9,11], it was shown that for $q \neq 1$, the convergence properties of the q-Bernstein polynomials are different from those of the classical ones. The approximation by a sequence of q-Bernstein polynomials with estimates for the rate of convergence in the case $q_n \uparrow 1$ was considered by Videnskii in [18–20], where he also studied modifications of these polynomials which improve the degree of approximation for $f \in C^2[0, 1]$.

In the case $q \in (0, 1)$, each q-Bernstein polynomial is a positive linear operator on $C[0, 1]$. Moreover, a sequence of q-Bernstein polynomials converges uniformly for each $f \in C[0, 1]$, and its limit is a positive linear operator on $C[0, 1]$, which we call the *limit q-Bernstein operator.* In distinction from the classical case $q = 1$, the sequence of q-Bernstein polynomials for $q \in (0, 1)$ does not satisfy the conditions of Korovkin's Theorem, and the limit q-Bernstein operator is *not* the identity operator. Recently, Wang [21] proved a general Korovkin-type theorem which is applicable to sequences of q-Bernstein polynomials. The theorem implies not only the existence of the limit q-Bernstein operator, but also an estimate for the rate of convergence, which is sharp for $f \in C^2[0, 1]$.

The approximation by the limit q-Bernstein operator as $q \uparrow 1$ was studied by Videnskii [18,19]. The connection of this operator with a generalized Poisson distribution is considered in [12]. Wang observed that the limit q-Bernstein operator arises as the limit of a sequence of q-Meyer-König and Zeller operators considered by Trif [16].

The limit q-Bernstein operator is a positive shape-preserving linear operator approximating continuous functions on [0, 1] as $q \uparrow 1$. Operators of this type are studied intensively in Approximation Theory (cf., e.g. [5]).

In this paper, we investigate the impact of the limit q-Bernstein operator on the analytic properties of functions. The change of smoothness under linear operators is an important problem of Classical Analysis. For instance, the smoothing of a function via convolution is used widely not only in Approximation Theory, but also in Distribution Theory, Fourier Analysis, and the Theory of Subharmonic Functions (cf., e.g. [6]). The usage of the smoothing method is based on the fact that, as a rule, the convolution operator improves the smoothness of a function. At the same time in some cases, the convolution causes a tremendous deterioration of smoothness (cf., e.g., [8]).

Our study reveals the following phenomenon: the limit q-Bernstein operator, in general, improves the analytic properties of functions. The improvement occurs for functions that are neither "too good" (polynomials) nor "too bad" (without the regularity condition given by (5.4)).

We apply the obtained results to finding eigenfunctions of the limit q-Bernstein operator. The eigenstructure of the classical Bernstein operator is described in [2], where the authors also demonstrate various applications of their results. The complete description of the spectrum and eigenfunctions of the limit q-Bernstein operator remains an open problem, for which some initial steps have been made in [11].

We use the following standard notation (cf., e.g. [1, Chapter 10, Section 10.2]):

$$(z; q)_0 := 1, \quad (z; q)_n := \prod_{k=0}^{n-1} (1 - zq^k), \quad (z; q)_\infty = \prod_{k=0}^{\infty} (1 - zq^k),$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} \text{ for } q \neq 1, \quad \begin{bmatrix} n \\ k \end{bmatrix}_1 := \binom{n}{k}, \quad 0 \leqslant k \leqslant n.$$

Definition 1.1 (*Philips [14]*). Let $f : [0, 1] \to \mathbf{C},\ q > 0$. The *q-Bernstein polynomial* of *f* is

$$B_n(f, q; x) = \sum_{k=0}^{n} f\left(\frac{1-q^k}{1-q^n}\right) p_{nk}(q; x), \quad n = 1, 2, \ldots ,$$

where

$$p_{nk}(q; x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (x; q)_{n-k}, \quad k = 0, 1, \ldots n.$$

To describe the behavior of $\{B_n(f, q; x)\}$ in the case $q \in (0, 1)$ and $n \to \infty$, consider the entire functions $p_{\infty k}(q; x) := \lim_{n\to\infty} p_{nk}(q; x)$, that is

$$p_{\infty k}(q; x) = \frac{x^k}{(q; q)_k} (x; q)_\infty, \quad k = 0, 1, \ldots . \tag{1.1}$$

By Euler's Identity (cf. [1, Chapter 10, Corollary 10.2.2]), we have

$$\sum_{k=0}^{\infty} p_{\infty k}(q; x) = 1 \text{ for all } x \in [0, 1). \tag{1.2}$$

For $f \in C[0, 1]$, we set:

$$B_\infty(f, q; x) := \begin{cases} \sum_{k=0}^{\infty} f(1-q^k) p_{\infty k}(q; x) & \text{if } x \in [0, 1), \\ f(1) & \text{if } x = 1. \end{cases} \tag{1.3}$$

Note that since the sequence $\{f(1-q^k)/(q, q)_k\}$ is bounded, $B_\infty(f, q; x)$ admits an analytic continuation into the unit disc $\{z : |z| < 1\}$. Whenever $B_\infty(f, q; x)$ admits an analytic continuation into a domain $D \subseteq \mathbf{C}$, we denote the continued function by $B_\infty(f, q; z),\ z \in D \subseteq \mathbf{C}$.

Theorem (*Il'inskii and Ostrovska [9]*). *For $q \in (0, 1)$ and any $f \in C[0, 1]$,*

$$B_n(f, q; x) \to B_\infty(f, q; x) \text{ as } n \to \infty \text{ uniformly on } [0, 1]. \tag{1.4}$$

The equality $B_\infty(f, q; x) = f(x)$ holds if and only if $f(x) = ax + b$.

As a result, for $q \in (0, 1)$ the sequence $\{B_n(f, q; x)\}$ is not approximating a function *f* unless *f* is linear. It should be noted here that this is completely in contrast to the case $q = 1$, where $\{B_n(f, 1; x)\}$ approximates *f* for any $f \in C[0, 1]$. For $q \in (1, \infty)$ approximating properties of $\{B_n(f, q; x)\}$ were investigated in [11].

Definition 1.2. Let $q \in (0, 1)$. The linear operator on $C[0, 1]$ given by

$$B_\infty = B_{\infty,q} : f \mapsto B_\infty(f, q; x)$$

is called the *limit q-Bernstein operator*.

The theorem above shows that this operator arises naturally when we consider the limit of a sequence of q-Bernstein polynomials for $q \in (0, 1)$.

It follows from (1.1)–(1.3) that B_∞ is a bounded positive linear operator on $C[0, 1]$ with $\|B_\infty\| = 1$. It is readily seen from (1.3) that B_∞ possesses the *end-point interpolation* property:

$$B_\infty(f, q; 0) = f(0),\ B_\infty(f, q; 1) = f(1) \text{ for all } q \in (0, 1). \tag{1.5}$$

The theorem of [9] above says that B_∞ leaves invariant linear functions. Besides (cf. [9, Theorem 1]), for any $f \in C[0, 1]$,

$$B_\infty(f, q; x) \to f(x) \text{ as } q \uparrow 1 \text{ uniformly on } [0, 1].$$

More information on approximation by B_∞ is given in [18,19].

In this paper, we discuss the connection between the smoothness of f and the analytic properties of its image $B_\infty(f, q; x)$ with $q \in (0, 1)$ being fixed. It has already been mentioned that for any $f \in C[0, 1]$, the function $B_\infty(f, q; z)$ is analytic in $\{z : |z| < 1\}$. In Section 2, we discuss the conditions for analytic continuation of $B_\infty(f, q; x)$ into a disc $\{z : |z| < R\}$, where $R > 1$. It is shown that the smoother f at 1 is, the greater R becomes; and if f is infinitely differentiable at 1, then $B_\infty(f, q; z)$ is entire. In Section 3, we give estimates of growth for this entire function via magnitudes of consecutive derivatives of f. These results imply that for any entire function f, the growth of $B_\infty(f, q; z)$ does not exceed the growth of $(z; q)_\infty$. In Section 4, we show that for an entire function whose growth is slower than that of $(z; q)_\infty$, this bound can be improved. Finally, in Section 5, we discuss images of functions with "bad" smoothness, that is, without the Lipschitz condition. We show that under certain regularity conditions, B_∞ speeds up the rate of $f(x)$ approaching $f(1)$ as $x \uparrow 1$. This is done in terms of the local modulus of continuity.

2. Images of Lipschitz continuous functions and differentiable functions

It was proved in [9] that for any $f \in C[0, 1]$, the function $B_\infty(f, q; x)$ is continuous on [0,1] and admits an analytic continuation into the unit disc $\{z : |z| < 1\}$. In general, $B_\infty(f, q; x)$ may not be analytically continued into a wider disc. For example, if $f \in C[0, 1]$ is such that $f(0) = f(1) = 0,\ f(1-q^k) = 1/k,\ k = 1, 2, \ldots$, then $B_\infty(f, q; x)$ is not differentiable at 1.

However, we will show that under some conditions concerning the smoothness of f in a left neighborhood of 1, $B_\infty(f, q; x)$ can be analytically extended into a disc of radius > 1. We need the next lemma, which shows a remarkable property of B_∞. Namely, this operator takes binomial $(1 - x)^j$ to the corresponding q-binomial $(1 - x)(1 - qx)\ldots(1 - q^{j-1}x)$.

Lemma 2.1. *The following identities hold*:

$$B_\infty\left((1-t)^j, q; z\right) = (z; q)_j, \quad j = 0, 1, 2, \ldots . \tag{2.1}$$

Corollary 2.2 (*Il'inskii and Ostrovska [9]*). *If f is a polynomial, then $B_\infty(f, q; z)$ is also a polynomial and* $\deg B_\infty(f, q; z) = \deg f$.

Proof of Lemma 2.1. It suffices to prove (2.1) for $x \in [0, 1)$.

By Definition (1.3) and Euler's Identity (cf. [1, Chapter 10, Corollary 10.2.2]), we have:

$$B_\infty\left((1-t)^j, q; x\right) = (x; q)_\infty \sum_{k=0}^{\infty} \frac{(q^j x)^k}{(q; q)_k} = \frac{(x; q)_\infty}{(q^j x; q)_\infty} = (x; q)_j. \qquad \square$$

The following theorem shows that the possibility of an analytic continuation for $B_\infty(f, q; x)$ is affected by the smoothness of f at 1 (which is, in fact, the smoothness along the sequence $\{1 - q^k\}$).

Theorem 2.3. (i) *If f is m times differentiable from the left at* 1, *then* $B_\infty(f, q; x)$ *admits analytic continuation into the disc* $\{z : |z| < q^{-m}\}$.

In particular, if f is infinitely differentiable from the left at 1, then $B_\infty(f, q; z)$ *is entire.*

(ii) *Let f have* $m \geqslant 0$ *derivatives at* 1 *and* $f^{(m)}(x)$ *satisfy the Lipschitz condition at 1, that is*

$$|f^{(m)}(x) - f^{(m)}(1)| \leqslant M(1-x)^\alpha \text{ for some } M > 0, \quad 0 < \alpha \leqslant 1. \tag{2.2}$$

Then $B_\infty(f, q; x)$ *admits an analytic continuation into the disc* $\{z : |z| < q^{-(m+\alpha)}\}$.

Remark 2.1. The results of Theorem 2.3 are sharp. In general, if *f* has exactly $m \geqslant 0$ derivatives at 1, then $B_\infty(f, q; x)$ may not be differentiable at $z = q^{-m}$, and hence $B_\infty(f, q; x)$ may not admit an analytic continuation into a disc of radius $R > q^{-m}$. Consider the following example:

Let $f \in C[0, 1]$ be so that

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{(1-x)^m}{\ln(1/(1-x))} & \text{if } x \in [1-q, 1), \\ 0 & \text{if } x = 1. \end{cases}$$

Clearly, *f* has *m* derivatives at 1 and $B_\infty(f, q; z)$ is analytic in $\{z : |z| < q^{-m}.\}$ Moreover, the following representation is valid:

$$\begin{aligned} B_\infty(f, q; z) &= (z; q)_\infty \sum_{k=0}^{\infty} \frac{f(1-q^k)}{(q; q)_k} z^k \\ &= (z; q)_\infty \sum_{k=1}^{\infty} \frac{(q^m z)^k}{k(q; q)_k \ln(1/q)}, \quad |z| < q^{-m}. \end{aligned}$$

We set $B_\infty(f, q; q^{-m}) = 0$ in order that the function be continuous on $[0, q^{-m}]$.

We show that $B_\infty(f, q; x)$ is not differentiable at q^{-m}, and therefore it cannot be analytically extended into a wider disc. Indeed, for $x \in (0, q^{-m})$,

$$\begin{aligned} \frac{B_\infty(f, q; x) - B_\infty(f, q; q^{-m})}{x - q^{-m}} &= -\frac{q^m (x; q)_\infty}{1 - q^m x} \sum_{k=1}^{\infty} \frac{(q^m z)^k}{k(q; q)_k \ln(1/q)} \\ &= -q^m (x; q)_m (x q^{m+1}; q)_\infty \sum_{k=1}^{\infty} \frac{(q^m z)^k}{k(q; q)_k \ln(1/q)}. \end{aligned}$$

We notice that

$$\sum_{k=1}^{\infty} \frac{(q^m z)^k}{k(q; q)_k \ln(1/q)} \geqslant \frac{1}{\ln(1/q)} \ln \frac{1}{1 - q^m x} \to \infty \text{ as } x \uparrow q^{-m}.$$

Since $q^m (q^{-m}; q)_m (q; q)_\infty \neq 0$, it follows that

$$\frac{B_\infty(f, q; x) - B_\infty(f, q; q^{-m})}{x - q^{-m}} \to \infty \text{ as } x \uparrow q^{-m}.$$

Thus, $B_\infty(f, q; x)$ is not differentiable at q^{-m}.

Similarly, for $0 < \alpha \leqslant 1$, consider

$$f_\alpha(x) = \begin{cases} (1-x)^\alpha & \text{if } 0 < \alpha < 1, \\ (1-x) \cos\left(\frac{\ln(1-x)}{\ln q}\right) & \text{if } \alpha = 1. \end{cases}$$

Clearly, $f_\alpha(x)$ satisfies the Lipschitz condition of order α at 1 (for $\alpha = 1$, we set $f(1) = 0$). Therefore, $B_\infty(f_\alpha, q; x)$ has an analytic continuation into a disc $\{z : |z| < q^{-\alpha}\}$ and it is not difficult to see that it cannot be analytically extended into a disc $\{z : |z| < R\}$, where $R > q^{-\alpha}$.

Proof of Theorem 2.3. (i) Suppose that f is m times differentiable from the left at 1. By Taylor's formula

$$f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(1)}{j!}(x-1)^j + r_m(x) =: T_m(x) + r_m(x),$$

where $T_m(x)$ is a polynomial and the remainder r_m is estimated by

$$r_m(x) = o\left((1-x)^m\right), \quad x \uparrow 1. \tag{2.3}$$

Obviously, $B_\infty(f, q; x) = B_\infty(T_m, q; x) + B_\infty(r_m, q; x)$. Since, by Corollary 2.2, $B_\infty(T_m, q; x)$ is a polynomial, it suffices to prove that $B_\infty(r_m, q; x)$ can be analytically continued into $\{z : |z| < q^{-m}\}$. Using (1.1) and (1.3), we get

$$B_\infty(r_m, q; z) = (z; q)_\infty \sum_{k=0}^{\infty} \frac{r_m(1-q^k)\, z^k}{(q; q)_k}, \quad |z| < 1, \tag{2.4}$$

where $(z; q)_\infty$ is an entire function and $\lim_{k\to\infty}(q; q)_k = (q; q)_\infty \neq 0$. Besides, we get from (2.3) that

$$|r_m(1-q^k)| \leqslant C_m q^{mk} \ \text{ for some } \ C_m > 0 \ \text{ and all } \ k = 0, 1, \ldots . \tag{2.5}$$

Hence, the series in (2.4) converges for all $z \in \{z : |z| < q^{-m}\}$.

(ii) To prove the statement, we replace (2.5) with

$$|r_m(1-q^k)| \leqslant C_m q^{(m+\alpha)k} \ \text{ for some } \ C_m > 0 \ \text{ and all } \ k = 0, 1, \ldots$$

and obtain convergence of the series in (2.4) for all $z \in \{z : |z| < q^{-(m+\alpha)}\}$. □

3. Image of an infinitely differentiable function

Theorem 2.3(i) asserts that if f is infinitely differentiable at 1, then $B_\infty(f, q; z)$ is entire. Using estimates for $f^{(j)}(x)$, $j = 0, 1, \ldots$, we can derive conclusions concerning the growth of $B_\infty(f, q; z)$. We denote

$$M(r; f) := \max_{|z| \leqslant r} |f(z)|.$$

In the sequel, the letter C (possibly with indices) denotes a positive constant, which may not be specified explicitly. The indices on C will either be a numbering (if more than one constant is involved) or indicate a dependence on certain parameters. Following this convention, we write $f(x) \asymp g(x)$ if $C_1 g(x) \leqslant f(x) \leqslant C_2 g(x)$.

The following statement is a key point in our reasoning.

Theorem 3.1. *Let* $f \in C[0,1] \cap C^{\infty}[a,1]$, $0 \leqslant a < 1$. *Suppose that* $\left|f^{(j)}(x)\right| \leqslant M_j$ *for* $x \in [a,1]$ *and all* $j = 1, 2, \ldots$ *with* $|f(x)| \leqslant M_0$ for $x \in [0,1]$. *We set*

$$S(n) := \sum_{j=0}^{n} \frac{M_j}{j!},$$

$$k_0 = k_{0,a} := \min \left\{ k \in \mathbf{Z}_+ : 1 - q^k \in [a,1] \right\}.$$

Then for $r > 1$ *the following estimate holds*:

$$M(r; B_\infty f) \leqslant C_a r^{k_0} S(n+1)(-r; q)_\infty,$$

where

$$n = n(r) = \left[\frac{\ln(2r)}{\ln(1/q)} \right]. \tag{3.1}$$

(By $[x]$ we denote the greatest integer not exceeding x.)

Corollary 3.2. *If* $\{M_n/n!\}$ *is increasing*, *then*

$$M(r; B_\infty f) \leqslant C_a r^{k_0} \frac{M_{n+1}}{n!} (-r; q)_\infty,$$

where $n = n(r)$ *is given by* (3.1).

Proof of Theorem 3.1. For any $n = 1, 2, \ldots$, we write $f(x) = T_n(x) + r_n(x)$, where

$$T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(1)}{j!}(x-1)^j.$$

Clearly, $|T_n(x)| \leqslant S(n)$ for $x \in [0,1]$. It follows from (2.1) that

$$|B_\infty(T_n, q; z)| \leqslant \sum_{j=0}^{n} \frac{M_j}{j!} |(z; q)_j| \leqslant \sum_{j=0}^{n} \frac{M_j}{j!} (-|z|; q)_j \leqslant S(n)(-|z|; q)_\infty. \tag{3.2}$$

Obviously, $|r_n(x)| \leqslant |f(x)| + |T_n(x)| \leqslant M_0 + S(n)$ for $x \in [0,1]$. Using (1.3), we write

$$\begin{aligned} B_\infty(r_n, q; z) &= (z; q)_\infty \sum_{k=0}^{k_0} r_n(1-q^k) \frac{z^k}{(q; q)_k} \\ &\quad + (z; q)_\infty \sum_{k=k_0+1}^{\infty} r_n(1-q^k) \frac{z^k}{(q; q)_k} \\ &=: (z; q)_\infty (\sigma_1 + \sigma_2). \end{aligned} \tag{3.3}$$

We have

$$|\sigma_1| \leqslant (M_0 + S(n)) \sum_{k=0}^{k_0} \frac{|z|^k}{(q; q)_k}.$$

Taking into account that $(q;q)_k > (q;q)_\infty$, we get

$$|\sigma_1| \leqslant (M_0 + S(n)) \, \frac{(k_0+1)|z|^{k_0}}{(q;q)_\infty} \quad \text{for} \quad |z| > 1. \tag{3.4}$$

To estimate σ_2, we notice that

$$r_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \, (x-1)^{n+1} \quad \text{where} \quad \xi \in (x, 1).$$

Therefore,

$$|r_n(1-q^k)| \leqslant \frac{M_{n+1}}{(n+1)!} \, q^{k(n+1)} \quad \text{for} \quad k > k_0.$$

It follows that

$$|\sigma_2| \leqslant \frac{M_{n+1}}{(n+1)!} \sum_{k=k_0+1}^{\infty} \frac{\left(q^{n+1}|z|\right)^k}{(q;q)_\infty}.$$

We fix z with $|z| > 1$ and choose n depending on z in such a way that $q^{n+1}|z| \leqslant 1/2$. We take

$$n = \left[\frac{\ln(2|z|)}{\ln(1/q)} \right]. \tag{3.5}$$

With this choice of n, we get

$$|\sigma_2| \leqslant \frac{1}{(q;q)_\infty} \frac{M_{n+1}}{(n+1)!} \frac{1}{2^{k_0}}. \tag{3.6}$$

Juxtaposing (3.3), (3.4), and (3.6), we get for $|z| > 1$,

$$\begin{aligned} |B_\infty(r_n, q; z)| &\leqslant |(z;q)_\infty| \, (|\sigma_1| + |\sigma_2|) \\ &\leqslant \frac{(-|z|;q)_\infty}{(q;q)_\infty} \left\{ (M_0 + S(n)) \, (k_0+1) \, |z|^{k_0} + \frac{M_{n+1}}{(n+1)!} \cdot \frac{1}{2^{k_0}} \right\} \\ &\leqslant \frac{(-|z|;q)_\infty S(n+1)}{(q;q)_\infty} \left\{ 2 \, (k_0+1) \, |z|^{k_0} + 2^{-k_0} \right\} \leqslant (-|z|;q)_\infty S(n+1) C_1 |z|^{k_0}. \end{aligned}$$

Hence, using (3.2), we get for $|z| > 1$,

$$|B_\infty(f, q; z)| \leqslant |B_\infty(T_n, q; z)| + |B_\infty(r_n, q; z)| \leqslant C_2 |z|^{k_0} S(n+1)(-|z|;q)_\infty,$$

where $C_2 = C_a$. $\square$

Now we will show that under relatively mild conditions for M_n, $B_\infty(f, q; z)$ is an entire function of a rather slow growth.

Theorem 3.3. (i) *If* $M_n \leqslant C_0 \exp\exp(\rho n)$ *for some* $\rho > 0$, *then*

$$M(r, B_\infty f) \leqslant C_1 \exp\left(C_2 \exp\left(\frac{\rho \ln r}{\ln(1/q)} \right) \right) \quad \text{for} \quad r \geqslant r_0, \tag{3.7}$$

that is, $B_\infty f$ *has finite order* $\leqslant \rho/(\ln(1/q))$.

(ii) *If* $M_n \leqslant C_0 \exp(n^p)$ *for some* $p > 0$, *then*

$$M(r, B_\infty f) \leqslant C_1 \exp\left(C_2 (\ln r)^{p_1}\right) \quad for \quad r \geqslant r_0, \tag{3.8}$$

where $p_1 = \max\{p, 2\}$, *that is* $B_\infty f$ *has finite logarithmic order* $\leqslant p_1$.

For functions analytic in $\{z : |z-1| \leqslant \tau < 1\}$, the Cauchy Theorem implies for $\delta = \tau/2$ that

$$f^{(n)}(x) = \frac{n!}{2\pi i} \oint_{|z-1|=\tau} \frac{f(\zeta)\, d\zeta}{(\zeta - x)^{n+1}}, \quad x \in [1-\delta, 1].$$

Hence, for $x \in [1-\delta, 1]$ we obtain:

$$\left| f^{(n)}(x) \right| \leqslant \frac{n! 2\pi\tau C_f}{2\pi\delta^{n+1}} = 2C_f n! \delta^{-n} \quad \text{with} \quad C_f = \max_{|\zeta-1|=\tau} |f(\zeta)|.$$

We set

$$M_n := C_1 n! \delta^{-n}, \quad C_1 = 2C_f, \; n = 0, 1, 2, \ldots .$$

By Stirling's Formula,

$$n! \asymp n^{n+\frac{1}{2}} e^{-n}.$$

Therefore,

$$M_n \leqslant C_2 \exp\left\{ \left(n + \frac{1}{2}\right) \ln n - n + n \ln(1/\delta) \right\} = O(\exp\{n \ln n + O(n)\}), \; n \to \infty.$$

Hence, Theorem 3.3 (ii) implies the following statement:

Corollary 3.4. *If f is analytic at* 1, *then*

$$M(r; B_\infty f) \leqslant C_1 \exp\left(C_2 \ln^2 r\right). \tag{3.9}$$

Remark 3.1. Estimate (3.9) is sharp. For example, if $f \in C^\infty[0,1]$ with $f(0) = 1$ and $f(x) = 0$, $x \in [1-q, 1]$, then by (1.3), $B_\infty(f; z) = (z; q)_\infty$ and

$$M(r; B_\infty f) = M(r; (z; q)_\infty) \geqslant C \exp\left\{ \frac{\ln^2 r}{2 \ln(1/q)} \right\}, \quad r \geqslant r_0.$$

Proof of Theorem 3.3. (i) We set $A_n := C_0 \exp\exp(\rho n)$. Obviously, $\{A_n/n!\}$ is increasing starting from some place, say, for $n > m$. Hence

$$S(n+1) = \sum_{j=0}^{n+1} \frac{M_j}{j!} \leqslant \sum_{j=0}^{n+1} \frac{A_j}{j!} = \sum_{j=0}^{m} + \sum_{j=m+1}^{n+1} \leqslant C + \frac{A_{n+1}}{n!}, \quad n > m.$$

Theorem 3.1 implies that for $r > 1$ and $n = n(r) > m$, we have:

$$\begin{aligned} M(r; B_\infty f) &\leqslant C_1 r^{k_0} \left(C + \frac{A_{n+1}}{n!}\right) (-r; q)_\infty \\ &\leqslant C_2 r^{k_0} \exp\exp\{\rho\,(n(r)+1)\}\,(-r; q)_\infty. \end{aligned}$$

Substituting (3.1) and taking into account that

$$(-r; q)_\infty \leqslant C \exp\left\{\frac{\ln^2 r}{2\ln(1/q)} + \frac{\ln r}{2}\right\}, \tag{3.10}$$

we get (3.7).

(ii) We set $A_n := C_0 \exp(n^{p_1})$. The sequence $\{A_n/n!\}$ is increasing starting from some place, say, for $n > m$. Hence, as in (i) we get

$$S(n+1) \leqslant C + \frac{A_{n+1}}{n!}, \quad n > m.$$

Therefore, by Theorem 3.1 we have

$$M(r; B_\infty f) \leqslant C_1 r^{k_0} \exp\left\{(n(r)+1)^{p_1}\right\}(-r; q)_\infty.$$

Substituting (3.1) and taking into account (3.10), we obtain (3.8). □

4. Image of an entire function

Corollary 3.4 implies that if f is entire, then $B_\infty(f, q; z)$ is an entire function with a rather slow growth restricted by (3.9). Since

$$B_\infty(f, q; z) = (z; q)_\infty \sum_{k=0}^{\infty} \frac{f(1-q^k)z^k}{(q; q)_k},$$

where $M(r; (z; q)_\infty) \geqslant C_1 \exp(C_2 \ln^2 r)$, it seems unlikely that estimate (3.9) can be essentially improved. However, for the entire functions f whose growth is slower than $\exp(C \ln^2 r)$, we can get a better estimate for $M(r; B_\infty f)$ than (3.9).

To do this, we need to express $B_\infty f$ using divided differences of f. For distinct points $x_0, x_1, \dots, x_k$, we denote by $f[x_0; x_1; \dots; x_k]$ divided differences of f, that is

$$f[x_0] = f(x_0), \quad f[x_0; x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \dots,$$

$$f[x_0; x_1; \dots; x_k] = \frac{f[x_1; \dots; x_k] - f[x_0; \dots; x_{k-1}]}{x_k - x_0}.$$

It is known (cf., e.g., [5, Chapter 4, Section 7, p.121]) that if x_i are all different, then

$$f[x_0; x_1; \dots; x_k] = \sum_{j=0}^{k} \frac{f(x_j)}{(x_j - x_0)\dots(x_j - x_{j-1})(x_j - x_{j+1})\dots(x_j - x_k)}. \tag{4.1}$$

Lemma 4.1. *For any $f \in C[0, 1]$, the following representation holds*:

$$B_\infty(f, q; z) = \sum_{k=0}^{\infty} q^{k(k-1)/2} f\left[0; 1-q; \dots; 1-q^k\right] z^k, \quad |z| < 1. \tag{4.2}$$

Remark 4.1. The representation remains true in any disc, where $B_\infty(f, q; z)$ admits an analytic continuation.

Proof of Lemma 4.1. By Eulers's Theorem (cf., e.g. [1, Chapter 10, Section 10.2]) we write:

$$(z;q)_\infty = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2}}{(q;q)_k} z^k.$$

Hence, by (1.1) and (1.3) we have for $|z| < 1$:

$$B_\infty(f,q;z) = \sum_{k=0}^{\infty} c_k z^k, \quad c_k = \sum_{j=0}^{k} \frac{f(1-q^j)}{(q;q)_j} \cdot \frac{(-1)^{k-j} q^{(k-j)(k-j-1)/2}}{(q;q)_{k-j}}.$$

We are to prove that $c_k = q^{k(k-1)/2} f\left[0; 1-q; \ldots ; 1-q^k\right], k = 0, 1, 2, \ldots$. Applying (4.1) we express the divided differences of f as follows:

$$f\left[0; 1-q; \ldots ; 1-q^k\right] = \sum_{j=0}^{k} \frac{(-1)^{k-j} f(1-q^j)}{q^{j(j-1)/2}(q;q)_j \, q^{j(k-j)}(q;q)_{k-j}}.$$

Multiplying by $q^{k(k-1)/2}$, we get c_k. □

Theorem 4.2. *Let f be a transcendental entire function. Then $B_\infty(f,q;z)$ is an entire function satisfying*

$$M(r; B_\infty f) = o(M(r; f)), \ r \to \infty. \tag{4.3}$$

Remark 4.2. If f is a polynomial of degree m, then Corollary 2.2 implies that

$$M(r; B_\infty f) \asymp M(r; f) \asymp r^m \ \text{ for } \ r \ \text{ large enough.}$$

We may apply Theorem 4.2 to finding the eigenstructure of B_∞. As a result, we get the following statement.

Corollary 4.3. *Let $B_\infty(f,q;x) = \lambda f(x), \ \lambda \neq 0, \ x \in [0, 1]$. If f satisfies the Lipschitz condition at* 1, *then f is a polynomial.*

Indeed, if f satisfies the Lipschitz condition at 1, then by Theorem 2.3(ii), $B_\infty(f; x)$ admits an analytic continuation into a disc of radius $R > 1$. Since $\lambda \neq 0$, the same is true for f. In particular, f is infinitely differentiable at 1. This, in turn, implies that $B_\infty f$ is an entire function, as well as f. By Theorem 4.2, f cannot be transcendental, because

$$M(r; B_\infty f) = |\lambda| M(r; f),$$

contrary to (4.3). Thus, f is a polynomial.

Remark 4.3. It has been shown in [11, Lemma 7], that for every $m = 0, 1, 2, \ldots$, the operator $B_{\infty,q}$ has an eigenvector $p_m(x)$ which is a monic polynomial of degree m, corresponding to the eigenvalue $q^{m(m-1)/2}$. For $m \geqslant 2$ such a polynomial is unique.

Proof of Theorem 4.2. Let f be a transcendental entire function. By ([10, Section 2.7, p. 44]) we have

$$f\left[0; 1-q; \ldots ; 1-q^k\right] = \frac{1}{2\pi i} \oint_L \frac{f(\zeta)\, d\zeta}{\zeta(\zeta-(1-q))\ldots(\zeta-(1-q^k))}, \tag{4.4}$$

where L is a contour around [0, 1].

If L is a circle of radius $r > 1$ centered at 0, then (4.4) implies

$$\left| f\left[0; 1-q; \ldots ; 1-q^k\right]\right| \leqslant \frac{1}{2\pi} \cdot \frac{2\pi r M(r; f)}{r(r-1)^k} = \frac{M(r; f)}{(r-1)^k}, \quad r > 1. \tag{4.5}$$

Now, given $\varepsilon > 0$, we choose N in such a way that

$$\sum_{k=N+1}^{\infty} q^{k(k-1)/2} 2^k < \varepsilon.$$

By (4.2)

$$(B_\infty f)(z) = \sum_{k=0}^{N} q^{k(k-1)/2} f\left[0; 1-q; \ldots ; 1-q^k\right] z^k + \sum_{k=N+1}^{\infty} q^{k(k-1)/2} f\left[0; 1-q; \ldots ; 1-q^k\right] z^k =: P_N(z) + R_N(z).$$

Hence

$$M(r; B_\infty f) \leqslant M(r; P_N) + \sum_{k=N+1}^{\infty} q^{k(k-1)/2} \left| f\left[0; 1-q; \ldots ; 1-q^k\right]\right| r^k.$$

For $r > 2$, we get using (4.5):

$$M(r; B_\infty f) \leqslant M(r; P_N) + M(r; f) \sum_{k=N+1}^{\infty} q^{k(k-1)/2} 2^k < M(r; P_N) + M(r; f) \cdot \varepsilon,$$

due to the choice of N. Since $M(r; P_N) = o(M(r; f)),\ r \to \infty$, and $\varepsilon > 0$ is arbitrary small, the statement follows. □

5. Image of a function with "bad" smoothness

Generally speaking, the theorems of Sections 2 and 3 imply that the operator B_∞ improves the analytic properties of a function f. In particular, B_∞ maps $C[0, 1]$ into a subset of $C[0, 1]$ consisting of functions that admit analytic continuation into the unit disc. Therefore, B_∞ improves possible "bad" smoothness of f on the half-open interval [0, 1).

Naturally, a question arises whether this operator improves possible "bad" smoothness at 1. Statement (i) of Theorem 2.3 shows that in some cases, this actually happens. Indeed, functions satisfying the Lipschitz condition at 1 are taken to functions analytic at 1.

Now, we will show that under certain additional conditions, the operator B_∞ speeds up the convergence of $f(x)$ to $f(1)$ as $x \uparrow 1$.

To measure the rate of $f(x)$ approaching $f(1)$ as $x \uparrow 1$, we use the local modulus of continuity at 1 defined by

$$\Omega(f;\delta) := \max_{1-\delta \leqslant x \leqslant 1} |f(x) - f(1)|. \tag{5.1}$$

Clearly, $\Omega(f;\delta)$ is a continuous monotone increasing function on [0, 1] with $\Omega(f;0) = 0$. Our further reasoning is based upon the following assertion.

Theorem 5.1. *For $f \in C[0,1]$, let $\Omega(f;\delta)$ be defined by* (5.1). *Then*

$$\Omega(B_\infty f;\delta) \leqslant C\delta \int_{q^{1/\delta}}^{1} \frac{\Omega(f;t)}{t}\,dt, \quad \delta \in (0, 1/2]. \tag{5.2}$$

Estimate (5.2) *is sharp up to a constant $C = C_{f,q}$.*

The following immediate corollaries of Theorem 5.1 illustrate an increase in the rate of $f(x)$ approaching $f(1)$ as $x \uparrow 1$.

Corollary 5.2. *If $\Omega(f;\delta) \leqslant C\,(\ln(1/\delta))^{-\alpha}$, $\alpha > 1$, $0 < \delta \leqslant \delta_0 < 1$, then for δ small enough, $\Omega(B_\infty f;\delta) \leqslant C_1\delta$, that is $B_\infty f$ satisfies the Lipshitz condition of order* 1 *at* 1.

Corollary 5.3. *If $\Omega(f;\delta) \leqslant C\,(\ln(1/\delta))^{-1}$, $0 < \delta \leqslant \delta_0 < 1$, then for δ small enough, $\Omega(B_\infty f;\delta) \leqslant C_1\delta\ln(1/\delta)$.*

Corollary 5.4. *If $\Omega(f;\delta) \leqslant C\,(\ln(1/\delta))^{-\alpha}$, $0 < \alpha < 1$, $0 < \delta \leqslant \delta_0 < 1$, then for δ small enough, $\Omega(B_\infty f;\delta) \leqslant C_1\delta^\alpha$, that is $B_\infty f$ satisfies the Lipshitz condition of order α at* 1.

Corollary 5.5. *If $\Omega(f;\delta) \leqslant C\,(\ln_k(1/\delta))^{-\alpha}$, $k > 1$, $\alpha > 0$, $0 < \delta \leqslant \delta_0 < 1$, then for δ small enough, $\Omega(B_\infty f;\delta) \leqslant C_1\,(\ln_{k-1} a/\delta)^{-\alpha}$, $a > 0$.*

Taking into account Corollary 4.3, we obtain the following assertion.

Corollary 5.6. *If $\Omega(f;\delta) \leqslant C\,(\ln_k(1/\delta))^{-\alpha}$, $k > 1$, $\alpha > 0$, $0 < \delta \leqslant \delta_0 < 1$, and $B_\infty(f, q; x) = \lambda f(x)$, $\lambda \neq 0$, $x \in [0,1]$, then f is a polynomial.*

Proof of Theorem 5.1. Since B_∞ reproduces linear functions and satisfies (1.5), we may assume without loss of generality that $f(1) = B_\infty(f, q; 1) = 0$. For $x \in [0, 1)$, we have

$$B_\infty(f, q; x) = (x;q)_\infty \sum_{k=0}^{\infty} f(1-q^k)\,\frac{x^k}{(q;q)_k}.$$

Obviously,

$$\frac{(x;q)_\infty}{(q;q)_k} \leqslant \frac{(1-x)(xq;q)_\infty}{(q;q)_\infty} \leqslant \frac{(1-x)}{(q;q)_\infty} =: C_q(1-x) \quad \text{for} \quad x \in [0,1).$$

Therefore,

$$|B_\infty(f,q;x)| \leqslant C_q(1-x)\sum_{k=0}^{\infty}|f(1-q^k)|x^k$$
$$\leqslant C_q(1-x)\sum_{k=0}^{\infty}\Omega(f;q^k)x^k, \quad x\in[0,1).$$

We set

$$w(y) := \Omega(f;q^y), \quad 0\leqslant y<\infty$$

and write the above estimate in the form

$$|B_\infty(f,q;x)|\leqslant C_q(1-x)\sum_{k=0}^{\infty}w(k)x^k.$$

Since $w(k)x^k$ is a non-increasing function in k with $x\in[0,1]$ being fixed, we get

$$\int_0^\infty w(y)x^y dy\leqslant \sum_{k=0}^{\infty}w(k)x^k\leqslant w(0)+\int_0^\infty w(y)x^y\,dy. \tag{5.3}$$

Taking $s=\ln(1/x)$, we estimate the integral in (5.3) as follows:

$$\int_0^\infty w(y)x^y dy = \int_0^\infty w(y)\exp(-ys)dy$$
$$= \int_0^{1/s} w(y)\exp(-ys)dy + \int_{1/s}^\infty w(y)\exp(-ys)dy$$
$$\leqslant \int_0^{1/s} w(y)\exp(-ys)dy + w(1/s)\frac{e^{-1}}{s}\leqslant 2\int_0^{1/s} w(y)\,dy.$$

Therefore,

$$|B_\infty(f,q;x)| \leqslant C_q(1-x)\left\{w(0)+2\int_0^{1/\ln(1/x)}w(y)\,dy\right\}$$
$$\leqslant C_q\left\{w(0)(1-x)+2\ln(1/x)\int_0^{1/\ln(1/x)}w(y)\,dy\right\}.$$

Due to the fact that $\ln(1/x)\int_0^{1/\ln(1/x)}w(y)\,dy$ is a decreasing function in x, we conclude that

$$\Omega(B_\infty f;\delta)\leqslant C_q\left\{w(0)\delta+2\ln\left(\frac{1}{1-\delta}\right)\int_0^{1/\ln(1/(1-\delta))}w(y)\,dy\right\}.$$

Recall that $w(y)=\Omega(f;q^y)$. Substituting $t=q^y$ into the last integral, we get

$$\Omega(B_\infty f;\delta)\leqslant C_q\left\{w(0)\delta+2\,\frac{\ln(1/(1-\delta))}{\ln(1/q)}\int_{q^{1/\ln(1/(1-\delta))}}^{1}\frac{\Omega(f;t)}{t}\,dt\right\}.$$

Since $\delta \leqslant \ln(1/(1-\delta)) \leqslant 2\delta$ for $\delta \in (0, 1/2]$, we get

$$\Omega(B_\infty f;\delta) \leqslant C_1\delta + C_2\delta \int_{q^{1/\delta}}^{1} \frac{\Omega(f;t)}{t}\,dt \quad \text{for} \quad \delta \in (0, 1/2]$$

and (5.2) now follows. We note that C_2 does not depend on f, while $C_1 = C_{q,f}$.

To show that (5.2) is sharp, we take $f(x) = \Omega(1-x)$, where $\Omega(\delta)$ is a continuous increasing function on [0, 1] with $\Omega(0) = 0$. In this case,

$$|B_\infty(f, q; x)| = \sum_{k=0}^{\infty} \Omega(q^k) p_{\infty k}(q; x).$$

We notice that for $x \in [0, 1]$,

$$p_{\infty k}(q; x) = \frac{x^k}{(q;q)_k}(x;q)_\infty \geqslant x^k(1-x)(q;q)_\infty =: C_q x^k(1-x).$$

Therefore, we get

$$\begin{aligned} |B_\infty(f, q; x)| &\geqslant C_q(1-x)\sum_{k=0}^{\infty}\Omega(q^k)x^k = C_q(1-x)\sum_{k=0}^{\infty} w(k)x^k \\ &\geqslant C_q(1-x)\int_0^\infty w(y)x^y\,dy, \end{aligned}$$

because of (5.3). This inequality implies that

$$\Omega(B_\infty f;\delta) \geqslant C_q\delta \int_0^\infty w(y)(1-\delta)^y\,dy.$$

We set $s = \ln(1/(1-\delta))$ and obtain for $\delta \in (0, 1/2]$:

$$\begin{aligned} \Omega(B_\infty f;\delta) &\geqslant C_q\delta\int_0^\infty w(y)\exp(-ys)dy \geqslant C_q\delta\int_0^{1/s} w(y)\exp(-ys)dy \\ &\geqslant C_q e^{-1}\delta \int_0^{1/s} w(y)dy = C\delta\int_0^{1/(\ln(1/(1-\delta))} w(y)dy \\ &\geqslant C\delta\int_{q^{1/\delta}}^{1}\frac{\Omega(t)}{t}\,dt. \qquad \square \end{aligned}$$

Theorem 5.1 along with Corollaries 5.2–5.5 show that in many cases the estimate for $\Omega(B_\infty f;\delta)$ is better than for $\Omega(f;\delta)$. This does not imply, however, that $\Omega(B_\infty f;\delta) = o\,(\Omega(f;\delta))$ as $\delta \downarrow 0$. For example, let $f = (1-x)^2$. Then, in virtue of Lemma 2.1, $B_\infty f = (1-x)(1-qx)$. Therefore, $\Omega(f;\delta) = \delta^2$, whereas $\Omega(B_\infty f;\delta) \asymp C\delta$.

The next theorem provides conditions for $\Omega(B_\infty f;\delta) = o\,(\Omega(f;\delta))$ as $\delta \downarrow 0$ to be true.

Theorem 5.7. *For $f \in C[0,1]$, let $\Omega(f;\delta)$ be defined by* (5.1). *We assume that $\Omega(f;\delta)$ satisfies the following regularity condition*:

$$\exists b \in (0,1), \quad \lim_{\delta\downarrow 0} \frac{\delta\int_{b^{1/\delta}}^{1}\frac{\Omega(f;t)}{t}\,dt}{\Omega(f;\delta)} = 0. \tag{5.4}$$

Then

$$\lim_{\delta\downarrow 0}\frac{\Omega(B_\infty f;\delta)}{\Omega(f;\delta)}=0. \tag{5.5}$$

Corollary 5.8. *If* $C_1\delta^\beta \leqslant \Omega(f;\delta)\leqslant C_2\,(\ln(1/\delta))^{-\alpha},\quad 0<\beta<\alpha<1$, *then* (5.5) *holds*.

Proof of Theorem 5.7. By virtue of (5.2), it suffices to prove that (5.4) implies

$$\lim_{\delta\downarrow 0}\frac{\delta\int_{q^{1/\delta}}^{1}\frac{\Omega(f;t)}{t}\,dt}{\Omega(f;\delta)}=0 \text{ for any } q\in(0,1). \tag{5.6}$$

If $q\geqslant b$, this is obvious.

If $q<b$, we are to prove that

$$\lim_{\delta\downarrow 0}\frac{\delta\int_{q^{1/\delta}}^{b^{1/\delta}}\frac{\Omega(f;t)}{t}\,dt}{\Omega(f;\delta)}=0.$$

We have:

$$\delta\int_{q^{1/\delta}}^{b^{1/\delta}}\frac{\Omega(f;t)}{t}\,dt \leqslant \delta\,\Omega(f;b^{1/\delta})\int_{q^{1/\delta}}^{b^{1/\delta}}\frac{dt}{t}$$
$$=\ln(b/q)\Omega\left(f;b^{1/\delta}\right)\leqslant\left(\frac{\ln(1/q)}{\ln(1/b)}-1\right)\delta\int_{b^{1/\delta}}^{1}\frac{\Omega(f;t)}{t}\,dt,$$

because

$$\delta\int_{b^{1/\delta}}^{1}\frac{\Omega(f;t)}{t}dt\geqslant\delta\,\Omega\left(f;b^{1/\delta}\right)\int_{b^{1/\delta}}^{1}\frac{dt}{t}=\ln(1/b)\Omega\left(f;b^{1/\delta}\right).$$

Therefore, (5.6) is true. □

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